Nicomachus on Means

In the preface to the first edition of my book, I heaped scorn on Nicomachus, frankly confessing that I found the idea of figurate numbers utterly useless. Since that time I have become more tolerant, having seen a few theorems (mentioned in the second edition) where they actually play a role. I therefore treated them in some detail in Chapter 7; however, I didn't have space to discuss the many kinds of means that Nicomachus introduces, nor their connection with music. Here is a brief sample of some of that material.

The last part of Nicomachus' treatise, comprising Chapters 22–29 of Book II, is devoted to the various kinds of ratios. It was mentioned above that these ratios were important in the Pythagorean theory of music and in their natural philosophy. Pythagoras had introduced three different kinds of means, the arithmetic, the geometric, and the harmonic. The first two of these are still very important today, and the third is by no means obsolete. Nicomachus was interested in what we now call progressions, where each term except the first and last is the appropriate mean of its predecessor and its successor. Thus, the sequence $5, 8, 11, 14, 17, 20$ is an arithmetic progression, since each term except the first and last is the average (arithmetic mean) of the term before it and the term after it. On the other hand $3, 15, 75, 375, 1875, 9375, 46875$ is a geometric progression, since each term is the mean proportional (geometric mean) of the term before it and the term after it. Nicomachus then makes an observation that, he says, most people had not noticed, namely that the square of each term in an arithmetic progression exceeds the product of its successor and its predecessor by exactly the square of the difference between successive terms. Thus $8^2 = 5 \cdot 11 + 3^2$. This fact would hardly have astounded the mathematicians of Mesopotamia 1500 years earlier, but to formulate it without our modern symbolic notation is still something of an achievement. Boethius credits Nicomachus with the discovery of this fact.

Harmonic progressions are no longer part of the high-school curriculum. They are more complicated than either of the other two. Nicomachus states their defining property as follows: *As the greatest term is to the smallest, so the difference between the greatest and mean terms is to the difference between the mean and smallest term.* In simpler terms, which Nicomachus might or might not have recognized, the reciprocals form an arithmetic progression. It is difficult to get a harmonic progression of any length that consists only of integers. For example, $3, 4, 6, 12$ is such a progression, but there is no way to extend it in either direction using only integers. A slightly longer one is $10, 12, 15, 20, 30, 60$, but again cannot be extended.

As with the figurate numbers, Nicomachus emphasizes that the point of all this theory of proportion is aesthetic. To play, for example, the chord C–G–C, the major fifth and the octave together, the string lengths required are in the ratio 6, 4, 3, a harmonic progression.[1]

Nicomachus goes on to develop a total of ten different kinds of means, with corresponding kinds of progressions associated with them, in which each term except the first and the last is the corresponding mean of its predecessor and its successor. He then proceeds to the summit of all this theory by introducing a type of four-term progression in which the first, third, and fourth terms form an arithmetic progression, while the first, second, and fourth form a harmonic progression. Such a progression will necessarily have something in common with a geometric progression, since the product of the first and last terms will equal the product of the two middle terms. An example, is the progression 4, 6, 8, 12. Nicomachus goes into raptures over this kind of progression, which he calls "most useful for all progress in music and in the theory of the nature of the universe." Indeed, he says, it really is the only true harmony:

> This alone would properly and truly be called harmony rather than the others, since it is not a plane, nor bound together by only one mean term, but with two, so as thus to be extended in three dimensions. . . [D'ooge, *1926*, 284–285].

[1] As modern physics looks at the situation, the frequencies, which are inversely proportional to the string lengths by the general theory of vibrating strings, are in the ratio 2, 3, 4, which is an arithmetic progression.

Literature

D'ooge, Martin Luther, 1926, translator. *Introduction to Arithmetic* (Nicomachus of Gerasa), Macmillan, New York.