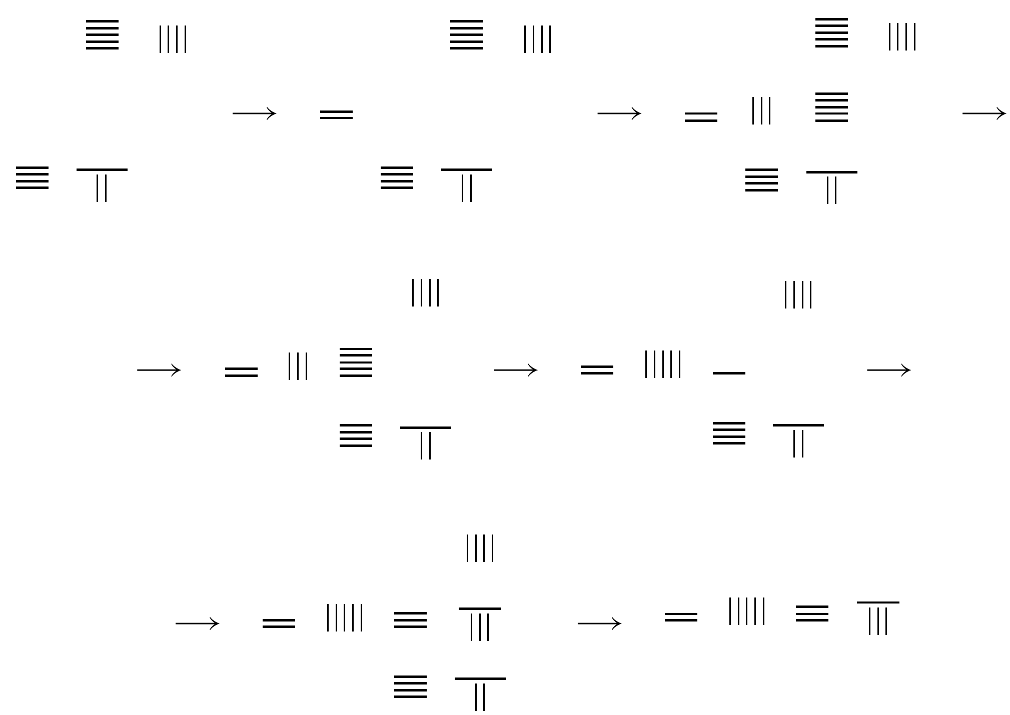
Multiplication with Shang Numerals

The text shows only what the Shang numerals are, but replaces them with Hindu-Arabic numerals in the discussion of computations. To give a better idea of what the actual procedure must have looked like, we offer the following pictorial representation of a simple multiplication.

The counting-board arrangement of the numbers is particularly well adapted to the alternating orientations that make it possible to keep track of the place-values of the Shang numerals. One simply uses the fixed larger number as a guide in placing the rods in their correct places in the partial products. As an illustration consider the following representation of the multiplication 54×47.



Fractions and Rational Numbers

According to Lam and Ang (*1992*, pp. 63–64), the development of the theory of fractions in China was closely allied to the establishment of the lunisolar calendar based on years of $12\frac{7}{19}$ months, equal to $365\frac{1}{4}$ days. Such a calendar is known to have been used during the Qin dynasty (221–201 BCE). Knowing how to calculate with fractions made it possible to calculate the length of a month in days:

$$365\frac{1}{4} \div 12\frac{7}{19} = \frac{1461}{4} \div \frac{235}{19} = \frac{1461 \times 19}{4 \times 235} = \frac{27759}{940} = 29\frac{499}{940}.$$

Roots

As discussed in the text, Chinese mathematicians dealt with roots of integers, some of which we now know to be irrational; and they found mixed numbers as approximations when the integer is not a perfect square. In the case of the example mentioned in the text, namely $\sqrt{355}$, the approximation would have been given as $18\frac{31}{36}$, as will be explained below. Here we shall give a more detailed discussion of the methods they used.

Let us begin by asking why anyone would want to find the square root of a number. The obvious answer, provided by geometry, is that we might want to know the side of a square whose area is given. But in what real-life situation would we want to know that? Here is a plausible scenario: An approximately rectangular plot of land 325 meters by 140 meters is to be divided into square lots and apportioned equally among 91 families. How long is the side of each family's square to be? Each family is to receive $325 \times 140 \div 91 = 500$ square meters of land. Hence we need to find $\sqrt{500}$. Such a scenario is at least possible without much distortion of normal human behavior. The *Sun Zi Suan Jing* presents the square root problem in this geometric form. Nevertheless, it should be kept in mind that the scenario is fictitious; many advances in mathematics arise simply because someone wondered about a problem and tried to find an answer.

The square root algorithm in general. We first explain the mathematics involved in taking a square root, so that the history of the procedure can be made comprehensible. When a number is given in decimal notation, its square root naturally is to be given the same way. Finding that square root proceeds one digit at a time. The first digit is found straightforwardly, after which a different procedure is applied repeatedly to find the following digits. Thus we have two procedures to describe.

The first step is comparatively easy, based on the fact that the number a has n digits, then a^2 has either $2n$ or $2n-1$ digits. Thus, for example, $31^2 = 961$ (three digits) while $34^2 = 1156$ (four digits). To apply this principle in reverse, say to get the first digit of the number 62854, we separate the digits from the right two at a time, writing the number as 6 28 54. To get the first digit then, we merely take the largest integer whose square is not larger than 6, namely 2. The square root will be somewhere between 200 and 300. This result can easily be checked since $200^2 = 40000 < 62854 < 90000 = 300^2$. Hence the first step involves merely knowing the square integers 1, 4, 9, 16, 25, 36, 49, 64, and 81. The left-most part of the number whose root is being taken will be either one or two digits, and the first digit of its square root will be the square root of the last number in this list of nine that is smaller than that one- or two-digit number.

The subsequent digits are obtained by following a procedure that can be modified for use when numbers are written in other ways. To explain it, suppose we are trying to find the square root of a number N and we have arrived at an approximation a to this square root. We define the error in the approximation to be $\varepsilon = N - a^2$, and we seek an adjustment h so that $a + h$ is a better approximation to N. If the approximation were exact, we would have

$$N = (a+h)^2 = a^2 + 2ah + h^2 = a^2 + h(2a+h),$$

so that

$$h = \frac{N - a^2}{2a+h} = \frac{\varepsilon}{2a+h}.$$

This equation is of no particular use in determining the *exact* square root, since h occurs on both sides. However, if we assume that a was already a fairly close approximation, then the adjustment h will be small, and so neglecting it on the right-hand side yields a good approximation for its value on the left-hand side:

$$h = \frac{\varepsilon}{2a}.$$

This equation provides the basis for obtaining the successive digits of the square root. To continue with our example of $\sqrt{62854}$, where our first approximation is $a = 200$, we find that $\varepsilon = 62854 - 40000 = 22854$, and $h = \frac{\varepsilon}{2a} = \frac{22854}{400} = 57.135$. Since we already have the hundreds digit in our first approximation, and we are finding only one digit at a time, we peel off the tens digit (5), thus taking effectively $h = 50$. We then have a new approximation $a + h = 250$, and we need to update the error, to get a new error $N - (a+h)^2 = N - a^2 - (2ah + h^2) = \varepsilon - (2a+h)h = 22854 - 22500 = 354$. This now becomes our new ε, and our new a is now 250. We then get a new h, namely

$$h = \frac{354}{500} = 0.708$$

We are now looking for the units digit, and this equation shows that it is zero. Hence we take $h = 0$ this time. There is then no need to update the value of a or ε. Hence for our next h we can go back to the same equation, this time taking the digit in the tenths position, namely 7. Our new approximation a is now 250.7, and we update the error again. The new error is $\varepsilon = 354 - 0.7(500 + 0.7) = 3.51$. The next h should be obtained as the hundredths digit of

$$\frac{3.51}{501.4} = 0.007000398883$$

Since this hundredths digit is zero, we can again leave the approximation and the error alone and get the thousandths digit from this expression. It is 7. In fact, at this point we are so close to the actual square root that we could take several more digits from this number without incurring any inaccuracy. (The actual square root is approximately 250.0707000301.)

The procedure just described works fairly smoothly. The only thing that can go wrong is that, neglecting h on one side of the equation $h = \frac{N-a^2}{2a+h}$ causes the new h on the other side of the equation to be *larger* than the exact value. Since we discard all of its digits after the first one we are interested in, the two approximation procedures tend to cancel each other's errors. But they might not. It may happen that h is larger than the true value by a sufficient amount that the next digit it reveals is too large. If that happens, the updated error will be negative, in which case it is necessary to reduce the digit by one and then recompute.

This whole process can be organized graphically for maximal efficiency in the way shown in the following example.

$$\begin{array}{ll}
 & \;\;2\;\;3\;\;5 \\
 & \sqrt{5\;52\;25} \\
 & \;\;4 \\
\hline
40 & 1\;52\;25 \\
43 & 1\;29 \\
\hline
460 & \;\;23\;25 \\
465 & \;\;23\;25
\end{array}$$

The Chinese version of the square root algorithm. The *Sun Zi Suan Jing* contains a description of the procedure for finding the square root of a positive integer. Necessarily, it mimics the procedure just described, in that it makes use of the numbers we have called a, $2a$, h, and ε at each stage. To illustrate it with the case of an exact square, we shall use an example from the book of Lam and Ang (*1992*), namely $\sqrt{700569}$. Certain mistakes or misprints in the method were corrected by the thirteenth-century mathematician Yang Hui, and the corrections are incorporated here.

$$\begin{array}{cccccc}
 & & & 8 & & \\
7 & 0 & 0 & 5 & 6 & 9 \\
 & & & & & \\
 & & & & & \\
 & 1 & & & &
\end{array}
\xrightarrow{(1)}
\begin{array}{cccccc}
 & & & 8 & & \\
7 & 0 & 0 & 5 & 6 & 9 \\
 & & & & & \\
 & 8 & & & & \\
 & 1 & & & &
\end{array}
\xrightarrow{(2)}
\left(\begin{array}{cccccc}
 & & & 8 & & \\
7 & 0 & 0 & 5 & 6 & 9 \\
 & & & & & \\
6 & 4 & & & & \\
 & 1 & & & &
\end{array}\right)$$

$$\xrightarrow{(3)}
\begin{array}{cccccc}
 & & & 8 & & \\
 & 6 & 0 & 5 & 6 & 9 \\
1 & 6 & & & & \\
 & & & & & \\
 & 1 & & & &
\end{array}
\xrightarrow{(4)}
\begin{array}{ccccc}
 & & 8 & & \\
6 & 0 & 5 & 6 & 9 \\
1 & 6 & & & \\
 & & & & \\
 & & 1 & &
\end{array}
\xrightarrow{(5)}
\begin{array}{ccccc}
 & & 8 & 3 & \\
6 & 0 & 5 & 6 & 9 \\
1 & 6 & & & \\
 & & 3 & & \\
 & & 1 & &
\end{array}$$

$$
\xrightarrow{(6)} \begin{pmatrix} & & 8 & 3 & 0 \\ 6 & 0 & 5 & 6 & 9 \\ 4 & 8 & & & \\ & & 9 & & \\ & & 1 & & \end{pmatrix}
\xrightarrow{(7)} \begin{array}{ccccc} & & 8 & 3 & \\ 1 & 1 & 6 & 6 & 9 \\ 1 & 6 & & & \\ & & 6 & & \\ & & 1 & & \end{array}
\xrightarrow{(8)} \begin{array}{ccccc} & & 8 & 3 & \\ 1 & 1 & 6 & 6 & 9 \\ 1 & 6 & 6 & & \\ & & & & \\ & & 1 & & \end{array}
$$

$$
\xrightarrow{(9)} \begin{array}{ccccc} & & 8 & 3 & \\ 1 & 1 & 6 & 6 & 9 \\ & 1 & 6 & 6 & \\ & & & & \\ & & & & 1 \end{array}
\xrightarrow{(10)} \begin{array}{ccccc} & & 8 & 3 & 7 \\ 1 & 1 & 6 & 6 & 9 \\ & 1 & 6 & 6 & \\ & & & & 7 \\ & & & & 1 \end{array}
\xrightarrow{(11)} \begin{pmatrix} & & 8 & 3 & 7 \\ 1 & 1 & 6 & 6 & 9 \\ 1 & 1 & 6 & 2 & \\ & & & 4 & 9 \\ & & & & 1 \end{pmatrix}
$$

$$
\xrightarrow{(12)} \begin{array}{cccc} & 8 & 3 & 7 \\ & & & 0 \\ 1 & 6 & 6 & \\ & & 1 & 4 \\ & & & 1 \end{array}
\xrightarrow{(13)} \begin{array}{cccc} & 8 & 3 & 7 \\ & & & 0 \\ 1 & 6 & 7 & 4 \\ & & & \\ & & & 1 \end{array}
$$

This seemingly complicated sequence of operations follows the successive-approximation pattern discussed above. We have $N = 700569$, and the first approximation to its square root is $a = 800$, obtained by pairing off the digits, as described above in the general procedure. We have suppressed the final zeros in the description, since on a counting board they would be represented by empty squares. The process of getting the next digit to adjoin to the given approximation a so as to produce a new a and updating the value of the error $\varepsilon = N - a^2$ requires five steps for the second digit and four for all subsequent steps. Thus in the display following the arrow numbered (4) all is in readiness for choosing the next digit (the new h), and h is chosen by dividing the error $N - a^2 = 60569$ by $2a = 1600$. As in the general discussion above, since the new digit will be a tens digit, we suppress the last two digits of $N - a^2 = 60569$ and the final digit of $2a = 1600$, dividing 605 by 160, to get the digit 3, representing 30.

It is easy to see that the top row of each display contains the successive approximations (a) to $\sqrt{N}$. Thus $a = 800$ through step (4), $a = 830$ in steps (5) through (9), and $a = 837$ in steps (10) through step (13). In the *Sun Zi Suan Jing* these numbers are called the *quotients* (*shang*). However, for the sake of consistency with the following rows, it is best to think of the first a as being 0 and the first approximation as h. That way, $2ah = 0$ until the second approximation has been chosen.

The second row contains the corresponding successive values of the error $\varepsilon = N - a^2$. Thus $\varepsilon = 700569$ through step (2), $\varepsilon = 700569 - (800)^2 = 60569$ from step (3) through step (6), $\varepsilon = 700569 - (830)^2 = 60659 - (2 \cdot 800 + 30)(30) = 11669$ from step (7) through step (11), and $700569 - (837)^2 = 11669 - (2 \cdot 830 + 7)7 = 0$. In the original these errors are called the *dividend* (*shi*).

The third row contains the current value of $2a$, which is 0 (meaning a blank row) until the second approximation is chosen. However, in the three stages at which the updating takes place, namely steps (2), (6), and (11)—which, it will be noted, are enclosed in parentheses—this row contains the current value of $2ah$ (the *square divisor* or *fang fa*).

The fourth row is empty except at the steps preceding an update, where it holds the current value of h, called the *side divisor* (*lian fa*) and at the update step (in parentheses), where it holds the current value of h^2. Hence to do the update of the error (second row), it is necessary to subtract from it the third and fourth rows

The fifth and last row is theoretically unnecessary, since the 1 in it, called the *lower divisor* (*xia fa*) has no numerical function. It is merely a bookmark for keeping track of where the algorithm is. The 1 moves two units to the right at each updating.

The question naturally arises as to how the Chinese handled square roots of non-square integers. Problem 19 of Chapter 2 in the *Sun Zi Suan Jing* asks for the side of a square whose area is 234567. Following the procedure above, we get the following sequence of displays:

$$
\begin{array}{cccccc}
 & & & 4 & & \\
2 & 3 & 4 & 5 & 6 & 7 \\
 & & & & & \\
 & & & & & \\
 & 1 & & & &
\end{array}
\xrightarrow{(1)}
\begin{array}{cccccc}
 & & & 4 & & \\
2 & 3 & 4 & 5 & 6 & 7 \\
 & & & & & \\
 & 4 & & & & \\
 & 1 & & & &
\end{array}
\xrightarrow{(2)}
\left(\begin{array}{cccccc}
 & & & 4 & & \\
2 & 3 & 4 & 5 & 6 & 7 \\
 & & & & & \\
1 & 6 & & & & \\
 & 1 & & & &
\end{array}\right)
$$

$$
\xrightarrow{(3)}
\begin{array}{ccccc}
 & & 4 & & \\
7 & 4 & 5 & 6 & 7 \\
8 & & & & \\
1 & & & &
\end{array}
\xrightarrow{(4)}
\begin{array}{ccccc}
 & & 4 & & \\
7 & 4 & 5 & 6 & 7 \\
 & 8 & & & \\
 & & & & \\
 & & 1 & &
\end{array}
\xrightarrow{(5)}
\begin{array}{ccccc}
 & & 4 & 8 & \\
7 & 4 & 5 & 6 & 7 \\
 & 8 & & & \\
 & & 8 & & \\
 & & & & \\
 & & 1 & &
\end{array}
$$

$$
\xrightarrow{(6)}
\left(\begin{array}{ccccc}
 & & 4 & 8 & \\
7 & 4 & 5 & 6 & 7 \\
6 & 4 & & & \\
 & 6 & 4 & & \\
 & & 1 & &
\end{array}\right)
\xrightarrow{(7)}
\begin{array}{cccc}
 & 4 & 8 & \\
4 & 1 & 6 & 7 \\
8 & & & \\
1 & 6 & & \\
 & & 1 &
\end{array}
\xrightarrow{(8)}
\begin{array}{cccc}
 & 4 & 8 & \\
4 & 1 & 6 & 7 \\
9 & 6 & & \\
 & & & \\
 & 1 & &
\end{array}
$$

$$
\xrightarrow{(9)}
\begin{array}{cccc}
 & 4 & 8 & \\
4 & 1 & 6 & 7 \\
 & 9 & 6 & \\
 & & & \\
 & & & 1
\end{array}
\xrightarrow{(10)}
\begin{array}{cccc}
 & 4 & 8 & 4 \\
4 & 1 & 6 & 7 \\
 & 9 & 6 & \\
 & & & 4 \\
 & & & 1
\end{array}
\xrightarrow{(11)}
\left(\begin{array}{cccc}
 & 4 & 8 & 4 \\
4 & 1 & 6 & 7 \\
3 & 8 & 4 & \\
 & & 1 & 6 \\
 & & & 1
\end{array}\right)
$$

$$
\xrightarrow{(12)}
\begin{array}{ccc}
4 & 8 & 4 \\
3 & 1 & 1 \\
9 & 6 & \\
 & & 8 \\
 & & 1
\end{array}
\xrightarrow{(13)}
\begin{array}{ccc}
4 & 8 & 4 \\
3 & 1 & 1 \\
9 & 6 & 8 \\
 & & \\
 & & 1
\end{array}
$$

Since the second row represents the error $\varepsilon = 234567 - (484)^2 = 311$, we see that the root lies between 484 and 485. What fraction is reasonable to take as the next and final approximation? The *Sun Zi Suan Jing* says the root should be taken as $484\frac{311}{968}$. Notice how easily the final result can be read directly from the final display! The numerator is the second row of the display and the denominator is the third row. But what makes this the most reasonable fraction to settle upon? We can explain it in our terms by noting that if $N = a^2 + \varepsilon$, then we can write $N = (a + h)^2$, where $h = \frac{\varepsilon}{2a+h} \approx \frac{\varepsilon}{2a}$. In the present case, $a = 484$, $\varepsilon = 311$, and $2a = 968$.

What can we learn about ancient Chinese mathematics from this square-root algorithm? Perhaps the most important thing is that the early Chinese mathematicians were adept at *thinking in terms of expressions*. Although they probably did not have a word corresponding to our word *variable* to denote an unspecified number, it would be impossible to construct such an algorithm as this without at least having a mental picture corresponding to the identity that we write as $N - (a + h)^2 = (N - a^2) - h(2a + h)$. It appears that they had formed a habit of thinking in terms of such expressions (sequences of operations), and had thereby begun to construct algebra. As discussed in Chapter 11, this square-root algorithm made it possible to solve what we consider to be quadratic equations. All that is necessary—the essential secret of doing algebra—is to think in terms of expressions and manipulate numbers accordingly. It seems likely that the Chinese were led to think this way from geometric considerations. The terms *square divisor* and *side divisor* strongly suggest that a "completing the square" picture lies in the background.

Literature

Lam Lay-Yong; Ang Tian-Se, 1992. *Fleeting Footsteps. Tracing the Conception of Arithmetic and Algebra in Ancient China*, World Scientific, River Edge, NJ.