Interpolation in Chinese Mathematics

Algebraic thinking is at the heart of interpolation, a skill that was once very useful in geometric astronomy, but has passed out of the curriculum now that tables of trigonometric functions and logarithms are no longer used. According to Li and Du (*1987*, pp. 88–90), a very sensitive and precise method of interpolation was used by the astronomer Liu Zhuo (544–610). Suppose the longitude of a planet has been observed at equal time intervals starting from time $t = 0, t = w, t = 2w, \ldots, t = nw$ and that the values of the longitude at these times are $L_0, L_1, \ldots, L_n$. Without "fitting" the data with a presumed formula relating L_k to kw, how can we obtain a reasonable estimate L_s of the longitude of the planet at a time between, say w and $2w$? The kind of linear interpolation usually done is based on the direct proportion

$$\frac{L_{1+s} - L_1}{L_2 - L_1} = \frac{s}{2-1} = s\,, \text{ that is, } L_{1+s} = L_1 + s\Delta\,,$$

where L_{1+s} is the estimate of the value at time $(1+s)w$, and $\Delta = L_2 - L_1$ is the increment in the longitude over the full time interval from w to $2w$. In terms of our concepts, this method amounts to joining the successive points $(0, L_0)$, $(w, L_1), \ldots, (kw, L_k), \ldots$ by straight lines. But linear interpolation is not very accurate when the planet accelerates or decelerates rapidly, as it does near its opposition. To get a better fit near these times, it is better to use three successive points instead of two and fit them with a parabola. For example, the parabola that passes through (w, L_1), $(2w, L_2)$, and $(3w, L_3)$ has the equation

$$y = \frac{L_2 - 2L_2 + L_1}{2w^2}x^2 - \frac{\frac{3}{2}L_3 - 4L_2 + \frac{5}{2}L_1}{w}x + (L_3 - 3L_2 + 3L_1)\,.$$

Its value at $x = (1+s)w$ can be arranged as

$$L_{1+s} = L_1 + s\Delta_1 + \frac{s(s-1)}{2}\Delta_1^2\,, \tag{1}$$

where $\Delta_1 = L_2 - L_1$ and $\Delta_1^2 = \Delta_2 - \Delta_2 = (L_3 - L_2) - (L_2 - L - 1) = L_3 - 2L_2 + L_1$. This procedure involves the kind of reasoning on unspecified numbers that we think of as algebra, even though no equations are being solved.

How could a formula or procedure corresponding to Eq. (1) be discovered? Apparently Liu Zhuo looked at the data on planetary observations at regular intervals over a long period of time and realized that the advances in longitude Δ were not equal. He therefore began to study the variation of these advances and found them to be much more regular, so regular that their differences could be regarded as constant. That is, the second differences $\Delta_k^2 = L_{k+2} - 2L_{k+1} - L_k$ are constant.* Obviously, knowing Δ_1, and $\Delta_1^2 = L_3 - 2L_2 + L_1$, it is possible to compute $\Delta_2 = \Delta_1 + \Delta_1^2$. But then since Δ_k^2 is the same for all k, it follows that $\Delta_k = \Delta_1 + (k-1)\Delta_1^2$. Then, having L_1 to start with, we get

$$\begin{aligned}
L_k &= L_1 + \Delta_1 + \cdots + \Delta_{k-1} \\
&= L_1 + (k-1)\Delta_1 + \sum_{j=2}^{k-1}(j-1)\Delta_1^2 \\
&= L_1 + (k-1)\Delta_1 + \frac{(k-1)(k-2)}{2}\Delta_1^2 \\
&= L_1 + (k-1)\Delta + \frac{(k-1)(k-2)}{2}\Delta^2\,.
\end{aligned}$$

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* Similar considerations may have motivated Ptolemy to formulate a law of refraction that is essentially quadratic.

where the subscript 1 can now be omitted, since only Δ_1 and Δ_1^2 occur. This last formula makes sense whether k is an integer or not and becomes Eq. (1) when $k = 1 + s$. Obviously since the second differences are not *exactly* constant, this last formula should not be extrapolated too far. The beauty of it is that one can start from any point as L_1.

Literature

Li Yan; Du Shiran, 1987. *Chinese Mathematics: A Concise History*, translated by John N. Crossley and Anthony W.-C. Lun, Clarendon Press, Oxford.