Heron's Formula for the Area of a Triangle

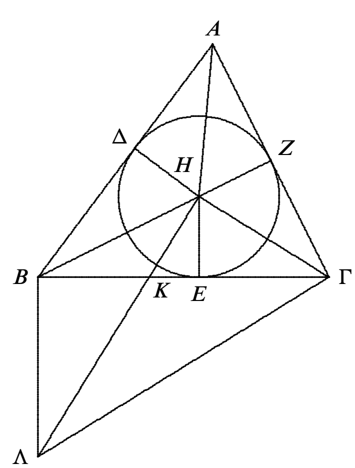
Although this topic is discussed in the text, space limitations precluded giving the details of the proof of the formula. We include here the complete proof of the formula.

Given any triangle $AB\Gamma$, inscribe a circle in it by the usual method of bisecting the angles. Let the center of the circle be H and its radius $HE = HZ = H\Delta$. Four simple observations need to be made. First, the angle bisectors divide the triangle into three triangles HAB, $HB\Gamma$, and $H\Gamma A$, all having the same altitude, namely the radius of the circle. Hence the area of the triangle is the radius of the circle times half the total length of the sides. Denoting this semiperimeter by Σ, we see that the triangle equals a rectangle whose sides are Σ and EH. Second, since the two right triangles at each vertex are congruent to each other, the semiperimeter Σ can be represented in many ways as a side plus one of the segments at the opposite vertex, for example $\Sigma = A\Delta + B\Gamma$. Third, this last fact means that, for example, $A\Delta = \Sigma - B\Gamma$, $BE = \Sigma - A\Gamma$, and $\Gamma E = \Sigma - AB$. Fourth, because of the congruence of the three pairs of right triangles, the six angles formed at H by the bisectors and the three radii HE, HZ, $H\Delta$ form three pairs of equal angles. In particular, the angle sum $\angle\Delta HA + \angle BH\Gamma$, being the sum of one angle from each pair, equals two right angles.

That being established, the proof is not long. Heron draws a perpendicular to $B\Gamma$ at B and a perpendicular to $H\Gamma$ at H, the latter intersecting $B\Gamma$ at K and the former at Λ, then connects $\Gamma\Lambda$. It follows immediately that triangle ΛBK is similar to triangle HKE. Also, because the triangles $\Lambda B\Gamma$ and $\Lambda H\Gamma$ are both right triangles having the same hypotenuse $\Lambda\Gamma$, the circle whose diameter is $\Lambda\Gamma$ passes through both B and H, that is, $\Lambda BH\Gamma$ is a cyclic quadrilateral. It follows that the sum of the opposite angles $B\Lambda\Gamma$ and $BH\Gamma$ is two right angles. But, by the fourth remark above, that means $\angle\Delta HA = \angle B\Lambda\Gamma$, and therefore triangle $\Lambda\Gamma B$ is similar to triangle $HA\Delta$. In particular $B\Gamma : B\Lambda :: \Delta A : \Delta H$, and so $B\Gamma : \Delta A :: B\Lambda : \Delta H :: B\Lambda : EH :: BK : KE$. Thus $B\Gamma : \Delta A :: BK : KE$ and so $(B\Gamma + \Delta A) : \Delta A :: (BK + KE) : KE :: BE : KE$. By the second and third remarks, this proportion can be written $\Sigma : (\Sigma - B\Gamma) :: (\Sigma - A\Gamma) : KE$. Since rectangles of the same width are proportional to their lengths, we can then write

$$\Sigma^2 : \Sigma\cdot(\Sigma - B\Gamma) :: \Sigma : (\Sigma - B\Gamma) :: (\Sigma - A\Gamma) : KE :: (\Sigma - A\Gamma)\cdot\Gamma E : KE\cdot\Gamma E$$
$$= (\Sigma - A\Gamma)\cdot(\Sigma - AB) : EH^2 .$$

Here $KE \cdot \Gamma E = EH^2$ because EH is the altitude to the base of the right triangle $HK\Gamma$.



Proof of Heron's method of computing the area of a triangle, probably due to Archimedes.